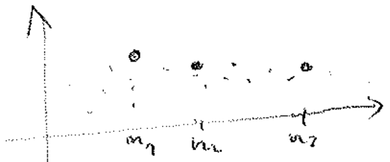
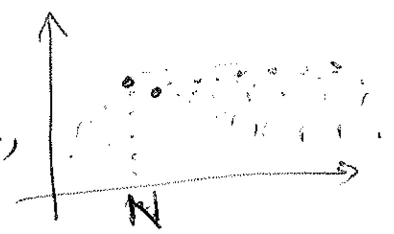
Th. Bolzano-Weierstrass theorem

Any bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Proof. $\{x_n\} \subset [a, b]$.



Lemma. $\{x_n\}$ has a monotone subsequence.
Let n be a "peak" ($x_n > x_m$ for all $m > n$). If there are infinitely many peaks, then $\{x_{n_1}, x_{n_2}, \dots\}$ is a monotone subsequence. If there are finitely many peaks, and N – last peak.
Then $N+1 =: n_1$ is not a peak, so there exists $n_2 > n_1$, $x_{n_2} \geq x_{n_1}$. But n_2 is not a peak, etc.,
... and thus $x_{n_1} \leq x_{n_2} \leq \dots$ as desired.



Hence there exists a convergent subsequence $\left(\lim_{i \to \infty} x_{n_i} = \inf_{i \in \mathbb{N}} (x_{n_i}) \text{ or } \sup_{i \in \mathbb{N}} (x_{n_i}) \right)$ □

Corollary: $[a, b]$ is sequentially compact.

Example. Prove, using the Borel-Lebesgue def. of compactness, that $[0, 1]$ is compact.

!

A (Heine-Borel: closed & bounded $\Rightarrow$ compact)

B (Sequential compactness: by Bolzano-Weierstrass)

C ~~Let $\mathcal{U} = \{U_i, i \in I\}$ be an open cover~~
we know that $[0, 1]$ is totally bounded, $E = \{\frac{1}{n}, \dots, \frac{n-1}{n}\}$ is an ε-net, with $n > \frac{1}{\varepsilon}$. $[0, 1]$ is also complete □

D Let's take an arbitrary open cover $\mathcal{U} = \{U_i, i \in I\}$. ~~For all $x \in [0,1]$, $x \in U_i$ for some i, and $\exists \varepsilon > 0$ $B_\varepsilon(x) \subset U_i$ because U_i is open~~
Consider $A = \{x \in [0, 1] : [0, x] \text{ covered by finitely many } U_i\text{'s}\}$. Let α be the least upper bound of A. We know that $\alpha \in [0, 1]$.
Suppose $\alpha < 1$. Then α in U_{i_0} and (open) so in some $B_\varepsilon(\alpha) \subset U_{i_0}$. But now $[0, \alpha - \frac{\varepsilon}{2}]$ is also covered by finitely many U_i's, $\{U_1, \dots, U_n\}$. Then $\{U_1, \dots, U_n\} \cup U_{i_0}$ covers $[0, \alpha + \frac{\varepsilon}{2}]$ which contradicts the def. of α. So $\alpha = 1$.

Example

$E = \{(u_n)_{n\in\mathbb{N}} : |u_n| \leq 1 \ \forall n \in \mathbb{N}\}$.

For all $u \in E$, let $\|u\| = \max\{|u_n|, n \geq 0\}$. Prove that there exists a seq. $(u(k)) \subset E$ without any convergent subsequence in $(E, \|\cdot\|)$, so E is not compact.

~~MODULUS~~ NORMS OF VECTORS DON'T CONVERGE IN ANY SUBSEQUENCE

~~Diagonal approach:~~

Not true in $C^m = I \times I \times \dots \times I$, $I = [-1, 1]$!

We would have that $\{u_1, u_2, \dots\} \subset C^m$ – compact. Hence sequentially compact.

It turns out that $C^\infty = I \times I \times \dots$ is not compact

Diagonal approach: ?

$$
\begin{aligned}
u(1) &= (u_1^1, u_2^1, u_3^1, u_4^1, \dots) \\
u(2) &= (u_1^2, u_2^2, u_3^2, u_4^2, \dots) \\
u(3) &= (u_1^3, u_2^3, u_3^3, u_4^3, \dots) \\
u(4) &= (u_1^4, u_2^4, u_3^4, u_4^4, \dots) \\
&\ \vdots
\end{aligned}
$$

$\Rightarrow$ Page 8

Theorem (Tychonoff). Product of any collection of compact topological spaces is compact (wrt to the induced product topology)

Corollary It holds true for metric spaces as well.

Warning: PRODUCT METRIC : $(X, d_1), (Y, d_2) \Rightarrow (X \times Y, d_\pi)$ with $d_\pi(z, z') = \sqrt{(d_1(x, x'))^2 + (d_2(y, y'))^2}$

Solution)

$d(u(i), u(j)) = \| u(i) - u(j) \| = \max_{k \in \mathbb{N}} | u_{ik} - u_{jk} |$ $\boxed{!}$

Let $u(i) = (0, \dots, 0, 1, 0, \dots) \in E$.
$\uparrow$ i-th place

We have $\| u(i) - u(j) \| = 1 \quad \forall i,j \in \mathbb{N}$. $(u(k))_{k \in \mathbb{N}}$ does not have a convergent subsequence.

Hence, the space $\ell_\infty([0,1])$ is <u>not</u> compact.
(Even though there is convergence coordinate-wise.)

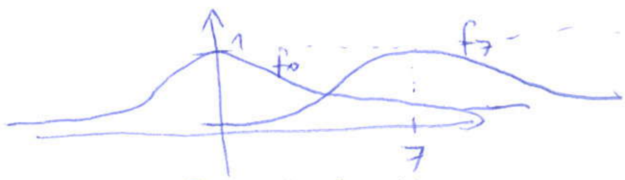
<u>Analogous example</u>

- Space $C[0,1]$ — space of bounded, continuous fcts $f: \mathbb{R} \to [0,1]$

The "sup" norm: $\| f \| = \sup_{x \in \mathbb{R}} | f(x) |$. [UNIFORM CONVERGENCE]

Hence $d(f_i, f_j) = \| f_i - f_j \| = \sup_{x \in \mathbb{R}} | f_i(x) - f_j(x) |$.

Let, e.g. $f_i(x) = e^{-(x-i)^2}$



$\forall x \in \mathbb{R} \ \lim_{n \to \infty} f_n(x) = 0$, <u>but</u> there is no uniform convergence as $\| f_n - 0 \| = 1$ for all n.

$\cancel{g(x) = f_i(x) - f_j(x)}$
$\cancel{g'(x) = -2(x-i)e^{-(x-i)^2}}$

If there exists a limit, it must be $f(x) \equiv 0$, but it turns out there's none. $\square$

Hence, $C[0,1]$ — not compact, ~~even though~~

Example

$$E = \{(u_n)_{n\in\mathbb{N}} : u_n \in \mathbb{R}\}$$

$$d(u,v) = \sum_{n\geq 0} \frac{\min\{1, |u_n - v_n|\}}{2^n}.$$

a) Well-defined distance (metric) fct.

1° $\forall (u, v \in E)$ $d(u,v) \geq 0$ and $d(u,v) \leq \sum_{n\geq 0} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2$

2° Axioms: $d(u,u) = \sum_{n\geq 0} \frac{\min\{1,0\}}{2^n} = 0.$

$$d(u,v) = \sum_{n\geq 0} \frac{\min\{1, |u_n - v_n|\}}{2^n} = d(v,u)$$

$$d(u,v) + d(v,z) = \sum_{n\geq 0} \left[\frac{\min\{1, |u_n - v_n|\} + \min\{1, |v_n - z_n|\}}{2^n}\right] =$$

$$\geq \sum_{n\geq 0} \left[\frac{\min\{2, 1+|u_n - v_n|, 1+|v_n - z_n|, |u_n - z_n|\}}{2^n}\right] \geq d(u,z).$$

"=" only if this case appears $\forall n \in \mathbb{N}$

b) $(u(k))$ — sequence of sequences $u \in E$

$(u(k)) \to u \iff (u_n(k))_{k\in\mathbb{N}} \to u_n$ for all n.

$$\boxed{\forall_{\varepsilon>0} \exists_{\bar{k}} \forall_{k>\bar{k}} \left(d(u(k), u) < \varepsilon\right) \iff \sum_{n\geq 0} \frac{\min\{1, |u_n(k) - u_n|\}}{2^n} < \varepsilon \iff \sum_{n\geq 0} \frac{|u_n(k) - u_n|}{2^n} < \varepsilon}$$

if valid!

$\Leftarrow$: $u_n(k) \xrightarrow[k\to\infty]{} u_n$ $\forall n$ then $\forall_{\varepsilon>0} \exists_{\tilde{k}} \forall_{k>\tilde{k}} |u_n(k) - u_n| < \frac{\varepsilon}{2}$

and so for $k > \max\{\bar{k}, \tilde{k}\}$, $\sum_{n\geq 0} \frac{|u_n(k) - u_n|}{2^n} < \frac{\varepsilon}{2} \sum \left(\frac{1}{2}\right)^n = \varepsilon.$

$\Rightarrow$: $\cancel{\sum_{n\geq N} \frac{|u_n(k) - u_n|}{2^n} < \varepsilon \Rightarrow \sum_{n\geq N} \frac{|u_n(k) - u_n|}{2^n} \leq \frac{|u_n(k) - u_n|}{2^n} < \varepsilon \Rightarrow |}$

for $k > \bar{k}$

ERRATUM (P. 9) 9A

Prove that $(u(k)) \subset E$ converges to $u \in E$, where (E, d), iff $(u_n(k))_{k\in\mathbb{N}} \xrightarrow[k\to\infty]{} u_n$ $\forall n \in \mathbb{N}$.

$\Rightarrow$: Assume that $\sum_{n\geq 0} \frac{\min\{1, |u_n - v_n|\}}{2^n} = d(u,v)$.

We have

$$\forall \varepsilon > 0 \ \exists \tilde{k} \ \forall k > \tilde{k} \quad \sum_{n\geq 0} \frac{\min\{1, |u_n(k) - u_n|\}}{2^n} < \varepsilon.$$

(By contradiction.) Assume furthermore that there exists $\bar{n} \in \mathbb{N}$ such that $\lim_{k\to\infty} |u_{\bar{n}}(k) - u_{\bar{n}}| > 0$. Call the distance $\delta > 0$.

Then $\sum_{n\geq 0} \frac{\min\{1, |u_n(k) - u_n|\}}{2^n} \geq \frac{\delta}{2^{\bar{n}}}$. Hence for $\varepsilon < \frac{\delta}{2^{\bar{n}}\cdot 2}$, for all $\tilde{k} \in \mathbb{N}$ there exist $k > \tilde{k}$ with $d(u(k), u) > \varepsilon$.

A contradiction. $\square$

$\Leftarrow$: Assume that $\forall n \in \mathbb{N}$, it is true that

$$\forall \varepsilon > 0 \ \exists \tilde{k}_{n,\varepsilon} \ \forall k > \tilde{k}_{n,\varepsilon} \quad |u_n(k) - u_n| < \frac{\varepsilon}{4}.$$

However, the convergence is <u>not uniform across n</u>. The conclusion is only true because we have the $\min\{1, |\cdot|\}$ part, and thus we only have to care about the initial <u>finite number</u> of coordinates. We have:

$$\forall \varepsilon > 0 \ \exists \bar{n} \ \forall n > \bar{n} \quad \sum_{m=n}^{+\infty} \frac{\min\{1, |u_m(k) - u_m|\}}{2^m} \leq \sum_{m=n}^{\infty} \left(\frac{1}{2}\right)^m = \left(\frac{1}{2}\right)^{n-1} \leq \left(\frac{1}{2}\right)^{\bar{n}-1}$$

Hence, for any $\varepsilon > 0$, we pick a sufficiently large <u>cutoff</u> point $\bar{n}$, such that $\left(\frac{1}{2}\right)^{\bar{n}-1} < \frac{\varepsilon}{2}$. This is always possible. Then, we have:

$$\forall \varepsilon > 0 \ \exists \tilde{k} \ \forall k > \tilde{k} \ d(u(k), u) \leq \sum_{m=0}^{\bar{n}} \frac{\min\{1, |u_m(k) - u_m|\}}{2^m} + \left(\frac{1}{2}\right)^{\bar{n}-1} \leq$$

$$\leq \sum_{m=0}^{\bar{n}} \frac{|u_m(k) - u_m|}{2^m} + \frac{\varepsilon}{2} \leq \max_{m=1,\dots,\bar{n}} |u_m(k) - u_m| \cdot 2 + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad \square$$

FINITE!!

c) Prove that (E, d) is compact.

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Let us prove that it is sequentially compact.

We know that $u(k) \xrightarrow[k\to\infty]{} u \iff \forall_{n\in\mathbb{N}}\ u_n(k) \xrightarrow[k\to\infty]{} u_n$.

→ Since for all $n \in \mathbb{N}$ $(u_n(k)) \subset \mathbb{R}$, by Bolzano-Weierstrass theorem we know that every $(u_n(k))$ has a convergent subsequence.

→ Take a sequence of sequences (infinite-dimensional vectors):

$(u(k)) = (u_1(k), u_2(k), u_3(k), \dots)$. We have to show that $(u(k))$ has a convergent subsequence.

→ Let $u_1(k_l)$ be convergent to u_1 (as $l \to \infty$).

Let $u_2(k_{l_m})$ — " — u_2 (as $m \to \infty$)

Let $u_3(k_{l_{m_p}})$ — " — u_3 (as $p \to \infty$)

$\vdots$

Let $u_{\bar{n}}(k_{l_{m_{p\dots q}}})$ — " — $u_{\bar{n}}$ (as $q \to \infty$). [Stop at a finite $\bar{n}$.]

→ We have (for $u = (u_1, u_2, u_3, \dots)$), for large $q > \tilde{q}$,

$$d(u(k_{l_{m_{p\dots q}}}) - u) \leq \sum_{n=0}^{\bar{n}} \frac{|u_n(k_{l\dots q}) - u_n|}{2^n} + \sum_{n=\bar{n}}^{+\infty} \left(\frac{1}{2}\right)^n.$$

Hence

$$\forall_{\varepsilon>0}\ \exists_{\bar{n}, \tilde{q}}\ \forall_{q>\tilde{q}}\ d(u(k_{l_{m\dots q}}) - u) \leq 2 \cdot \max_{n=1,\dots,\bar{n}} \overbrace{|u_n(k_{l\dots q}) - u_n|}^{<\frac{\varepsilon}{4}} + \overbrace{\left(\frac{1}{2}\right)^{\bar{n}-1}}^{<\frac{\varepsilon}{2}} < \varepsilon.$$

We have constructed a convergent subsequence.

Hence (E, d) is (sequentially) compact. $\square$

Unit ball in l_∞?

$\|u\| = \max_n \{|u_n| : n \in \mathbb{N}\}$

$$B_1(0) = \{u \in l_\infty : \|u\| \leq 1\} = \{u \in l_\infty : \max_{n \in \mathbb{N}} |u_n| \leq 1\} =$$

$$= \{(u_1, u_2, \dots) : \forall i \, |u_i| \leq 1\}.$$

Unit ball in (E, d)?

$$B_1(0) = \{u \in E : d(u, 0) \leq 1\} = \{u \in E : \sum_{n \geq 0} \frac{\min\{1, |u_n|\}}{2^n} \leq 1\}$$

E.g. $(1, 1, 0, 0, \dots) \notin B_1(0)$ because $1 + \frac{1}{2} = \frac{3}{2} > 1$

E.g. $(0, 0, 7, 1567, (1567)^2, \dots) \in B_1(0)$ because $\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2} < 1$

In finite dimensional spaces, $B_1(0)$ is bounded & closed and thus compact.

In infinite dimensional spaces,

Def. $B_1(0) = \{x \in X : \|x\| \leq 1\}$. Norm: $\|x\| = 0 \Leftrightarrow x = 0$, $\|\alpha x\| = |\alpha| \cdot \|x\|$, Triangle inequality

1° Since we are dealing with a vector space, there exists a basis $\mathcal{B} = \{x_i\}_{i \in I}$ such that $\forall x \in X \; \exists \{\alpha_i\}_{i \in I} \; x = \sum_{i \in I} \alpha_i x_i$.

2° Infinite dimensional $\Leftrightarrow$ the basis is not a finite set.

3° Take a sequence of $(x_i)_{i \in J}$, J - countable subset of I.

4° For all $i \in J$, $\|x_i\| > 0$ (by 1st axiom). Moreover, there exists $\alpha_i > 0$ such that $\|\alpha_i x_i\| = \alpha_i \|x_i\| = 1$ (by 2nd axiom).

5° So we have an infinite sequence $(a_i x_i)_{i \in J} \subset B_1(0)$.

6° This sequence does not have a convergent subsequence because $\{x_i\}_{i \in J} \subset \mathcal{B}$, they don't repeat and $\|\alpha_i x_i\|$ is bounded away from zero $\square$

Problem

A - compact in $(X,d) \Rightarrow A$ - complete

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A - seq. compact $\Rightarrow$ every $(x_n) \subset A$ has a convergent subsequence.

Then every Cauchy sequence has a conv. subsequence as well. $\{x_{n_k}\} \to \bar{x}$

$\forall \varepsilon > 0 \; \exists \tilde{n} \; \forall_{n, n_k \geq \tilde{n}} \; d(x_n, x_{n_k}) < \frac{\varepsilon}{2}$ and also $d(x_{n_k}, \bar{x}) < \frac{\varepsilon}{2}$

and thus $d(x_n, x_{n_k}) + d(x_{n_k}, \bar{x}) < \varepsilon$.

$d(x_n, \bar{x}) \leq$

Problem A - compact

$\{A_n\}$ - decreasing seq of closed sets, $A_{n+1} \subseteq A_n$.

$\rightarrow \bigcup_{n=1}^{\infty} A_n = A_1$ is obviously non-empty.

$\rightarrow x \in \bigcap_{n=1}^{\infty} A_n \Leftrightarrow \forall_n (x \in A_n)$.

?

SKIP THIS

A_n are bounded and closed $\forall n$, & nonempty.

Simpler examples $(\mathbb{C}(\mathbb{R}, [0,1]), \|\cdot\|)$ - ~~complete?~~ closed?

$\|f\| = \sup_{x \in \mathbb{R}} |f(x)|$.

CLOSED?

seq. (f_n) $\quad \|f_n - f\| \xrightarrow{n \to \infty} 0 \Leftrightarrow \sup_{x \in \mathbb{R}} |f_n(x) - f(x)| \to 0, \quad |f_n(x) - f(x)| < \frac{\varepsilon}{3}$.

f_n - continuous for all x, and thus $|x - x_0| < \delta \Rightarrow |f_n(x) - f_n(x_0)| < \frac{\varepsilon}{3}$.

$|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)| < \varepsilon$.

CLOSED.

~~Complete? $\|f_n - f_m\| \leq \varepsilon$~~

$\circled{14}$

$x_0 \geq 0$ given

$$\max_{\substack{x_t \in [0, f(x_{t-1})] \\ \forall t \geq 1}} \sum_{t \geq 1} \beta^t u(\overbrace{f(x_{t-1}) - x_t}^{c_t}) \qquad c_t + x_t = f(x_{t-1})$$

- $f: \mathbb{R}_+ \to \mathbb{R}_+$, bounded, continuous
  $u: \mathbb{R}_+ \to \mathbb{R}_+$ — " —

$$E = \{(x_t)_{t \in \mathbb{N}}\} \text{ with } d(x,y) = \sum_{n \geq 0} \frac{\min\{1, |x_n - y_n|\}}{2^n}$$

[ONLY THE BEGINNING MATTERS]

a) $V((x_t)_{t \in \mathbb{N}})$, $V: E \to \mathbb{R}$ — Let $V(x) = \sum_{t \geq 1} \beta^t u(f(x_{t-1}) - x_t)$!!

(E,d) compact

Let $d(x,y) < \delta$. ~~We get~~ This means that for all $n \in \mathbb{N}$, $|x_n - y_n| < \tilde{\delta}$. (particular property of (E,d))

Hence $|V((x_t)) - V((y_t))| = |\sum_{t \geq 1} \beta^t (u(f(x_{t-1}) - x_t) - u(f(y_{t-1}) - y_t))|$

By continuity of f, u,
$|f(x_{t-1}) - f(y_{t-1})| < \frac{\varepsilon}{7}$, etc.,

$|V(x_t) - V(y_t)| \leq |\sum_{t \geq 1}^{\infty} \beta^t \frac{\varepsilon}{7}| = \frac{\varepsilon}{7} \cdot \frac{\beta}{1-\beta}$. V-continuous.

b) $\forall t \; (0 \leq x_t \leq f(x_{t-1}))$

The set is: $\begin{cases} 0 \leq x_1 \leq f(x_0) \\ 0 \leq x_2 \leq f(x_1) \text{ etc.} \end{cases}$

$A \subset E$

- Any Intersection of closed sets is closed. So A is closed.
- $A \subset E$ and A is a closed subset of a compact space, so it's compact
- It is also bounded : $E \subset B_{45}(0)$ ...

c) <u>WEIERSTRASS</u>

Example (backward induction)

$$\max_{c_{0,2}} \sum_{t=0}^{2} \ln c_t \quad \text{s.t.} \quad S_{t+1} = (S_t - c_t)(1+r), \ r>0$$

S_0 - given, $S_t \geq 0$ for all t, $c_t \leq S_t$.

Bellman eq's.

$$V(S_0, 0; S_2, 2) = \max_{c_0} \{ \ln c_0 + V(S_1; 1; S_2; 2) \}$$
$$V(S_1, 1; S_2, 2) = \max_{c_1} \{ \ln c_1 + V(S_2; 2; S_2, 2) \}$$
$$V(S_2, 2; S_2, 2) = \max_{c_2} \{ \ln c_2 \}.$$

Solve backwards:

at $t=2$ $\max_{c_2} \{\ln c_2\}$ s.t. $c_2 \leq S_2$. Hence $c_2^*(S_2) = S_2$.

This implies the value $V(S_2, 2; S_2, 2) = \ln S_2 = \ln\left[(S_1 - c_1)(1+r)\right]$

At $t=1$ $\max_{c_1} \{ \ln c_1 + \ln((S_1 - c_1)(1+r)) \}$.

$$\frac{\partial}{\partial c_1}: \quad \frac{1}{c_1} + \frac{-1}{S_1 - c_1} = 0 \Rightarrow c_1 = S_1 - c_1 \Rightarrow c_1 = \tfrac{1}{2} S_1.$$

Policy fct: $c_1^*(S_1) = \frac{1}{2} S_1$. Value fct $V(S_1, 1; S_1; 2) =$
$= \ln(\frac{1}{2} S_1) + \ln(\frac{1}{2} S_1) + \ln(1+r)$.

At $t=0$ $\max_{c_0} \{ \ln c_0 + 2\ln \frac{1}{2}(S_0 - c_0)(1+r) + \ln(1+r) \} =$
$$= \max_{c_0} \{ \ln c_0 + 2 \ln \tfrac{1}{2}(S_0 - c_0) + 3 \ln(1+r) \}$$

$$\frac{\partial}{\partial c_0}: \quad \frac{1}{c_0} + 2 \frac{-1}{S_0 - c_0} = 0 \Rightarrow c_0 = \frac{S_0 - c_0}{2} \Rightarrow c_0 = \tfrac{1}{3} S_0.$$

Policy fct: $c_0^*(S_0) = \frac{1}{3} S_0$.
Value fct $V(S_0, 0; S_2, 2) = \ln \frac{1}{3} S_0 + 2 \ln \frac{1}{2} S_1 + \ln(1+r) =$
$= \ln \frac{1}{3} S_0 + 2 \ln \frac{1}{3} S_0 + 3 \ln(1+r) = 3 \ln(\frac{1}{3} S_0) + 3 \ln(1+r)$.

Optimal sequence satisfies: $c_0^*(S_0) = \frac{1}{3} S_0$, $c_1 = \frac{1}{3} S_0 (1+r)$, $c_2 = \frac{1}{3} S_0 (1+r)^2$

S_0; $S_1 = \frac{2}{3} S_0 (1+r)$, $S_2 = \frac{1}{3} S_0 (1+r)^2$.

Theorem (Envelope theorem)

Let $F(\alpha) = \max_{x \in X} f(x, \alpha)$, where f-differentiable, X-open

Then $\frac{d}{d\alpha} F(\alpha) = \frac{\partial f}{\partial \alpha}(x, \alpha)$.

Proof.

$$\frac{d}{d\alpha} F(\alpha) = \frac{d}{d\alpha} f(\overbrace{x^*(\alpha)}^{x^*(\alpha) = \arg\max_{x \in X} f(x,\alpha)}, \alpha) = \underbrace{\frac{\partial f}{\partial x}(x^*(\alpha), \alpha)}_{0} \cdot \frac{\partial x^*(\alpha)}{\partial \alpha} + \frac{\partial f}{\partial \alpha}(x, \alpha) \quad \square$$

- If the assumptions of DP don't hold, then — in finite time dimensions — we can still solve using the Lagrange method or Kuhn-Tucker.

So the infinite dimension is more interesting here...

Example (Fisheries / Forest management).

We have $\forall t$ $\underbrace{S_{t+1} = A \cdot S_t(\bar{S} - S_t) + S_t - C_t}$,

assume $S_t \in [0, \bar{S}]$ for all t, $C_t \leq S_t$, $\underbrace{0 \leq S_0 \leq \bar{S} \text{ given.}}$

We maximize the yield $u(c_t)$, discounted with $\beta \in (0,1)$

$$\max_{c_{t,\infty}} \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t) \right\} = V(s_0, 0) = \max_{c_0} \left\{ u(c_0) + \sum_{t=1}^{\infty} \beta^t u(c_t) \right\} =$$

$\tau = t-1$

one could expand $\beta u(c_1) + \ldots$

$$= \max_{c_0} \left\{ u(c_0) + \beta \sum_{\tau=0}^{\infty} \beta^\tau u(c_{\tau+1}) \right\}$$

$$= \max_{c_0} \left\{ u(c_0) + \beta V(s_1, 1) \right\}.$$

Analogously,

$$V(s_t, t) = \max_{c_t} \left\{ u(c_t) + \beta V(s_{t+1}, t+1) \right\}.$$

Existence of the value fct will be shown later!

TOMÁŠ.

Solution steps:

1° In the $\{\}$:

$$\frac{\partial}{\partial c_t}: \quad u'(c_t) + \beta V'(s_{t+1}, t+1) \cdot \frac{\partial m_t}{\partial c_t} = 0$$

$$u'(c_t) + \beta V'(s_{t+1}, t+1) \cdot (-1) = 0$$

$$\underbrace{u'(c_t) = \beta V'(s_{t+1}, t+1)}$$

2° Use the envelope theorem

$$\frac{\partial}{\partial s_t}: \quad V'(s_t) = \cancel{\max_{c_t}} \left\{ \beta V'(s_{t+1}, t+1) \cdot (1 + A(\bar{s} - 2s_t)) \right.$$

3° Put together: $u'(c_t) = V'(s_t, t) \cdot \frac{1}{1 + A(\bar{s} - 2s_t)}$

4° Shift by 1 period: $u'(c_{t+1}) = \frac{V'(s_{t+1}, t+1)}{1 + A(\bar{s} - 2s_{t+1})}$

5° Insert again to get rid of V':

$$\boxed{u'(c_t) = \beta u'(c_{t+1})(1 + A(\bar{s} - 2s_{t+1}))}$$

← Euler eq.

E.g. if $u(c_t) = \frac{c_t^{1-\theta} - 1}{1-\theta}$, $\theta > 0$, $\theta \neq 1$, (CRRA)

then $u'(c_t) = c_t^{-\theta}$,

$$c_t^{-\theta} = \beta c_{t+1}^{-\theta} (1 + A(\bar{s} - 2s_{t+1}))$$

$$\left(\frac{c_{t+1}}{c_t}\right)^{\theta} = \beta(1 + A(\bar{s} - 2s_{t+1}))$$

Assume that $(c_t, s_t) \xrightarrow{t \to \infty} \underbrace{(\tilde{c}, \tilde{s})}_{\text{STEADY STATE}}$ — we will show later that they will!

Then @SS, $c_{t+1} = c_t = \tilde{c}$, $s_{t+1} = s_t = \tilde{s}$,

The steady state satisfies

$$1 = \beta(1 + A(\bar{s} - 2\tilde{s}))$$

$$\Rightarrow \frac{1}{\beta} - 1 = A(\bar{s} - 2\tilde{s})$$

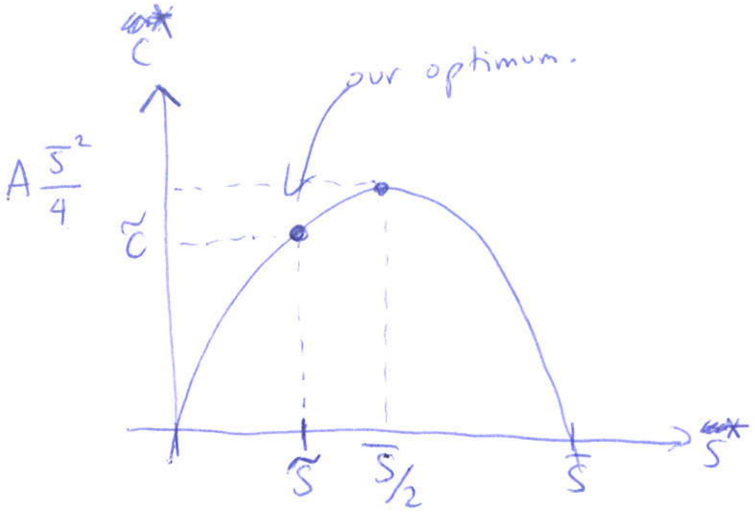
$$\frac{1-\beta}{A\beta} = \bar{s} - 2\tilde{s}$$

$$\boxed{\tilde{s} = \frac{1}{2}\left[\bar{s} - \frac{1-\beta}{A\beta}\right]} < \frac{1}{2}\bar{s}$$

$$\tilde{c} = A\tilde{s}(\bar{s} - \tilde{s}) =$$

$$= A \frac{1}{2}\left[\bar{s} - \frac{1-\beta}{A\beta}\right] \frac{1}{2}\left[\bar{s} + \frac{1-\beta}{A\beta}\right]$$

$$= \frac{A}{4}\left(\bar{s}^2 - \left(\frac{1-\beta}{A\beta}\right)^2\right).$$



at s.s. $S_{t+1} = S_t$,

So $C^*(S^*) = A S^*(\bar{S} - S^*)$

<u>Why?</u> Because we discount the future. We consume <u>more</u> <u>now</u>, and so in the long run our yield $\boxed{\tilde{C}}$ is <u>less</u> than the highest long-run possible yield $\boxed{C^*}$

$$\frac{\partial \tilde{S}}{\partial \beta} = -\frac{1}{2}\left[\frac{-A\beta - A(1-\beta)}{(A\beta)^2}\right] = +\frac{1}{2}\frac{1}{A\beta^2} > 0$$

As $\beta \to 1_-$, $\tilde{S} \to \frac{1}{2}\bar{S}$, and thus $\tilde{C} \to \frac{A}{4}\bar{S}^2$.

More generally, <u>discounting</u> allows one to write

$$V(x_t, t) = \max_{u_t, \dots} \sum_{s=t}^{T-1} \underbrace{\alpha_s F_s(u_s, x_s)}_{f_s(u_s, x_s)}$$

Inserting to the Bellman eq.,

$$V(x_t, t) = \max_{u_t} \left\{ \alpha_t F_t(u_t, x_t) + V(x_{t+1}, t+1) \right\}$$

In current units $(/\alpha_t)$

β_t - DISCOUNT FACTOR.

$$V^c(x_t, t) = \max_{u_t} \left\{ F_t(u_t, x_t) + \frac{\alpha_{t+1}}{\alpha_t} V^c(x_{t+1}, t+1) \right\}$$